SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

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4021. *Proposed by Arkady Alt.*

Let $(\overline{a}_n)_{n\geq 0}$ be a sequence of Fibonacci vectors defined recursively by $\overline{a}_0 = \overline{a}, \overline{a}_1 = \overline{b}$ and $\overline{a}_{n+1} = \overline{a}_n + \overline{a}_{n-1}$ for all integers $n \geq 1$. Prove that, for all integers $n \geq 1$, the sum of vectors $\overline{a}_0 + \overline{a}_1 + \cdots + \overline{a}_{4n+1}$ equals $k\overline{a}_i$ for some i and constant k.

We received nine correct solutions. We present the solution by David Stone and John Hawkins (joint).

We shall prove that $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1} = L_{2n+1}\bar{a}_{2n+2}$, where L_k is the kth Lucas number. We use some easily proven results. Here, F_k is the kth Fibonacci number.

1. $F_0 + F_1 + \cdots + F_m = F_{m+2} - 1$.
2. $F_{4n+2} = L_{2n+1}F_{2n+1}$
3. $F_{4n+3} = L_{2n+1}F_{2n+2} + 1$
4. $\bar{a}_k = F_{k-1}\bar{a}_0 + F_k\bar{a}_1$ for $k \geq 1$.

Therefore,

$$\begin{aligned}
\sum_{k=0}^{m} \bar{a}_k &= \bar{a}_0 + \sum_{k=1}^{m} (F_{k-1}\bar{a}_0 + F_k\bar{a}_1) \\
&= \bar{a}_0 + \left(\sum_{k=1}^{m} F_{k-1}\right)\bar{a}_0 + \left(\sum_{k=1}^{m}\right)\bar{a}_1 \\
&= \bar{a}_0 + (F_{m+1} - 1)\bar{a}_0 + (F_{m+2} - 1)\bar{a}_1 \\
&= F_{m+1}\bar{a}_0 + F_{m+2}\bar{a}_1 - \bar{a}_1 \\
&= \bar{a}_{m+2} - \bar{a}_1
\end{aligned}$$

Hence, with $m = 4n + 1$,

$$\begin{aligned}
\sum_{k=0}^{4n+1} \bar{a}_k = \bar{a}_{4n+3} - \bar{a}_1 &= F_{4n+2}\bar{a}_0 + F_{4n+3}\bar{a}_1 - \bar{a}_1 \\
&= (L_{2n+1}F_{2n+1})\bar{a}_0 + (L_{2n+1}F_{2n+2})\bar{a}_1 \\
&= L_{2n+1}(F_{2n+1}\bar{a}_0 + F_{2n+2}\bar{a}_1) \\
&= L_{2n+1}\bar{a}_{2n+2}.
\end{aligned}$$